Analysis 2

12 March 2024

Warm-up: Calculate $\int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$ for the function $f(x, y) = xy^3$ and …

… $x = -\cos(t)$, $y = \sin(t)$ with $\frac{\pi}{4} \leq t \leq \pi$.

if your surname starts A-F.

… $x = \ln(t^5)$, $y = \ln(t)$ with $1 \leq t \leq e$.

if surname G-R.

… $x = t$, $y = \sqrt{1 - t^2}$ with $\frac{-1}{\sqrt{2}} \leq t \leq 1$.

if surname S-Ż.

Warm-up: Calculate $\int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$ for the function $f(x, y) = xy^3$ and three different $\vec{r} : [a, b] \to \mathbb{R}^2$.

Your simplified integral should be

A–F: $\int_{\pi/4}^{\pi} (-\cos t)(\sin t)^3 \, dt = \ldots$

G–R: $\int_1^e \frac{5\sqrt{26}(\ln t)^4}{t} \, dt = \ldots$

S–Ž: $\int_{-1/\sqrt{2}}^{1} (t - t^3) \, dt = \ldots$

Warm-up: Calculate $\int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,dt$ for the function $f(x,y)=xy^3$ and three different $\vec{r}:[a,b]\to\mathbb{R}^2$.

Your simplified integral should be

A-F: $\int_{\pi/4}^{\pi}(-\cos t)(\sin t)^3\,dt = \frac{-1}{4}(\sin t)^4\Big|_{t=\pi/4}^{t=\pi} = \boxed{\frac{1}{16}}$

G-R: $\int_1^e \frac{5\sqrt{26}(\ln t)^4}{t}\,dt = \sqrt{26}(\ln t)^5\Big|_{t=1}^{t=e} = \boxed{\sqrt{26}}$

S-Ż: $\int_{-1/\sqrt{2}}^{1}(t-t^3)\,dt = \left(\frac{1}{2}t^2-\frac{1}{4}t^4\right)\Big|_{t=-1/\sqrt{2}}^{t=1} = \boxed{\frac{1}{16}}$

Last week

	a scalar (number) as output	a vector (or multiple numbers) as output
a scalar (number) as input		"vector function"
a vector (or multiple numbers) as input	"scalar function" or "scalar field"	"vector field"

Last week

	a scalar (number) as output	a vector (or multiple numbers) as output
a scalar (number) as input	$f(x)$ $x(t)$ $P(x)$	$\vec{r}(t)$
a vector (or multiple numbers) as input	$f(x, y, z)$ $T(x, t)$	$\vec{F}(x, y, z)$

Last week

A **scalar function** is a function whose output is a number but whose input can be thought of as a list of numbers or as a single vector. We often write

$$f(x, y)$$

for a function

$$f : \mathbb{R}^2 \to \mathbb{R}.$$

and $f(x, y, z)$ for $f : \mathbb{R}^3 \to \mathbb{R}$.

A curve (path) in 2D or 3D can be described using *parametric equations* or using a single *vector equation*. Therefore a vector function

$$\vec{r} : [a, b] \to \mathbb{R}^n$$

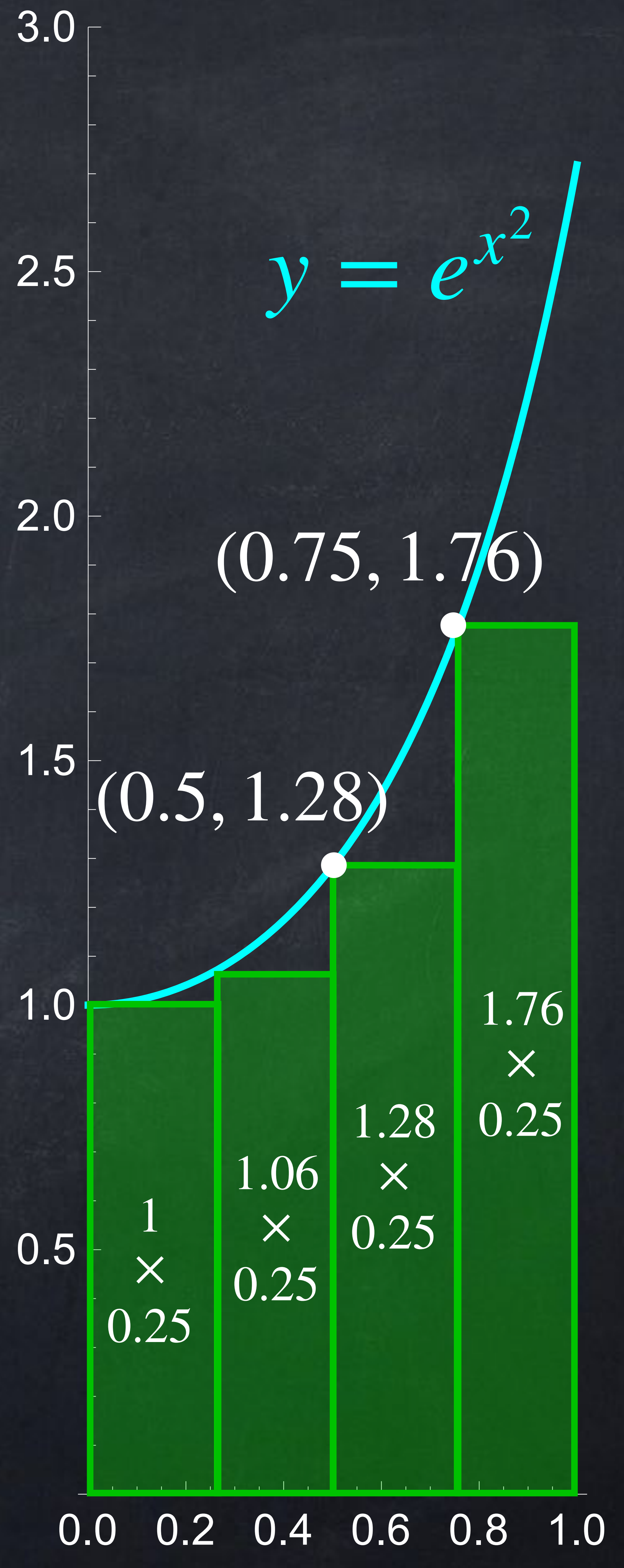
can also describe a curve.

Integrals

Is $\int_0^1 e^{x^2}\,\mathrm{d}x$ more or less than 1?

- more or less than 2?
- more or less than 3?

How could we get an approximate value for this integral?



$0.25 + 1.06\times0.25 + 1.28\times0.25 + 1.76\times0.25$

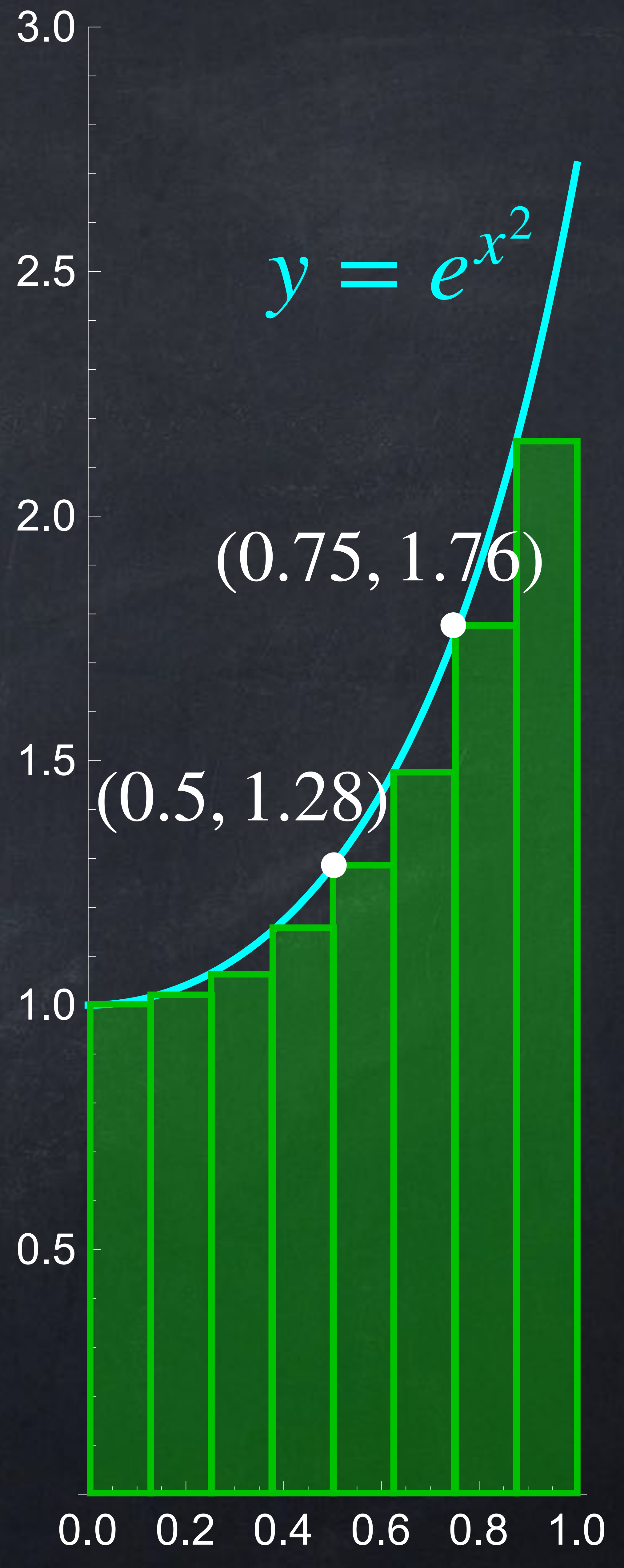
$= 0.25 + 0.2661 + 0.321 + 0.439$

$= 1.276$

Integrals

Is $\int_0^1 e^{x^2} \mathrm{d}x$ more or less than 1?

- more or less than 2?
- more or less than 3?



How could we get an approximate value for this integral?

$0.125 + 1.015 \times 0.125 + \cdots + 1.76 \times 0.125 + 2.15 \times 0.125$

$= 0.125 + 0.127 + \cdots + 0.219 + 0.269$

$= 1.362$

We can approximate $\int_a^b f(x)\mathrm{d}x$ by adding up several terms that are

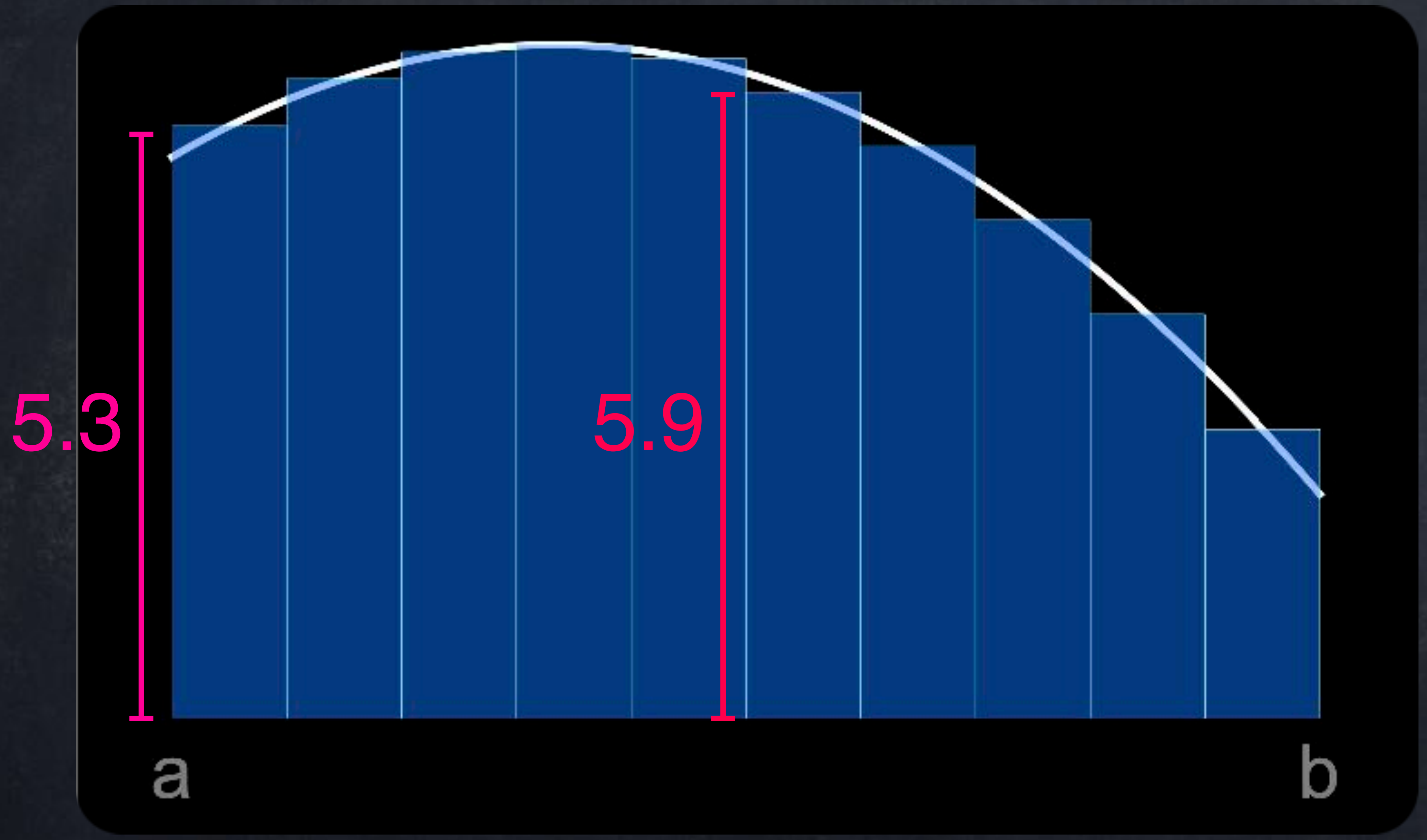
(f value) × (length of a small interval).

without actually drawing any rectangles.

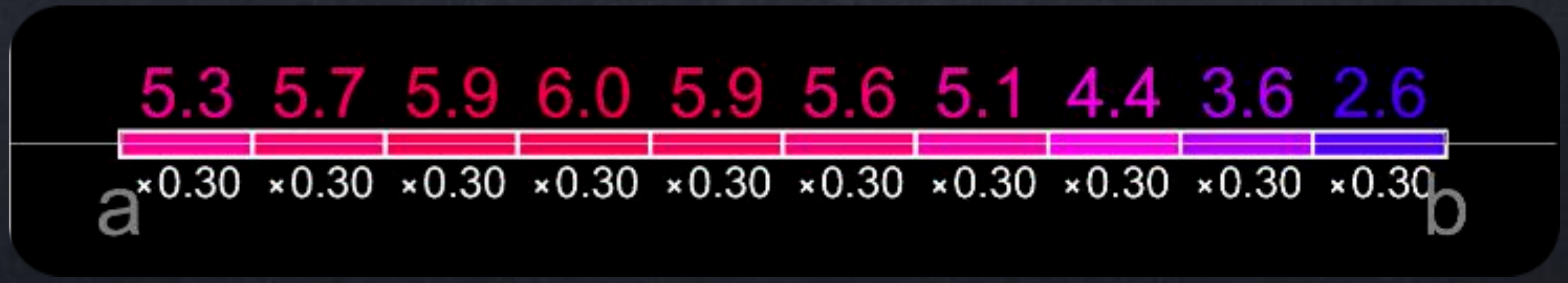
- Officially, the integrals is *defined* as the limit of this kind of sum.
- The Fundamental Theorem of Calculus tells us that we can use anti-derivatives to calculate integrals instead (if we can find a formula for the anti-derivative of $f(x)$).

The $\sum f \cdot \mathrm{length}$ idea lets us draw a 1D picture instead of a 2D picture…

Analysis 1: $\int_a^b f(x)dx$



Area

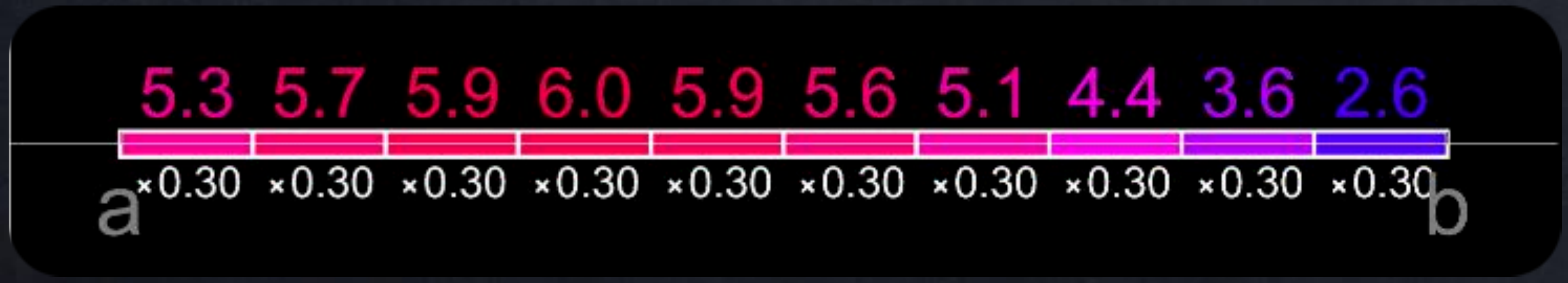


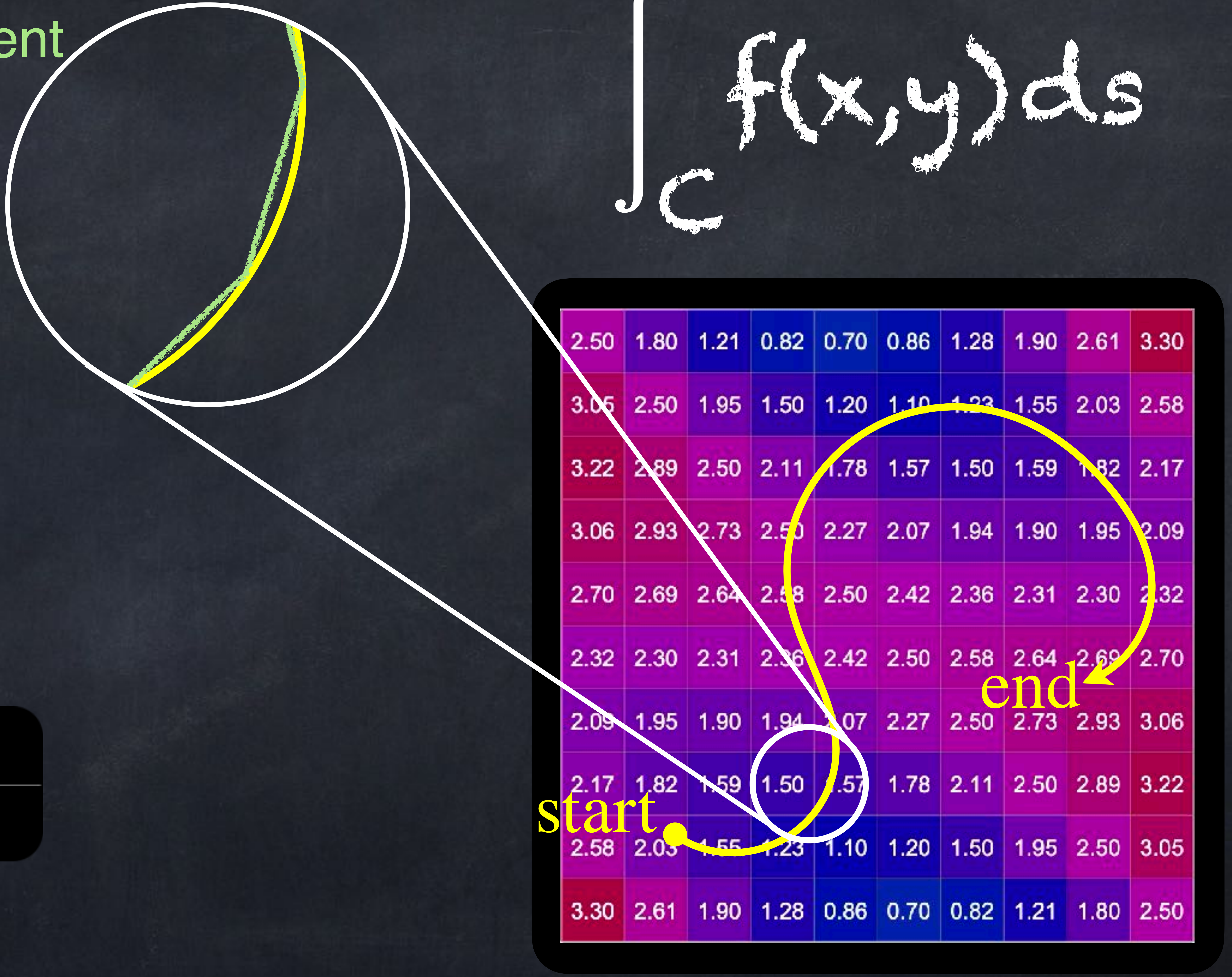
Anything

The length of each small line segment is exactly $\sqrt{(x')^2 + (y')^2}$, so that's why its in our path-integral formula.

$$\int_C f(x,y)\,ds$$

$$\int_a^b f(x)\,dx$$





The **path integral** of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ over a curve C is written as

$$\int_C f \, \mathrm{d}s.$$

Officially, this is the limit of a sum of f-values multiplied by lengths of small intervals (small line segments connecting points on the curve C).

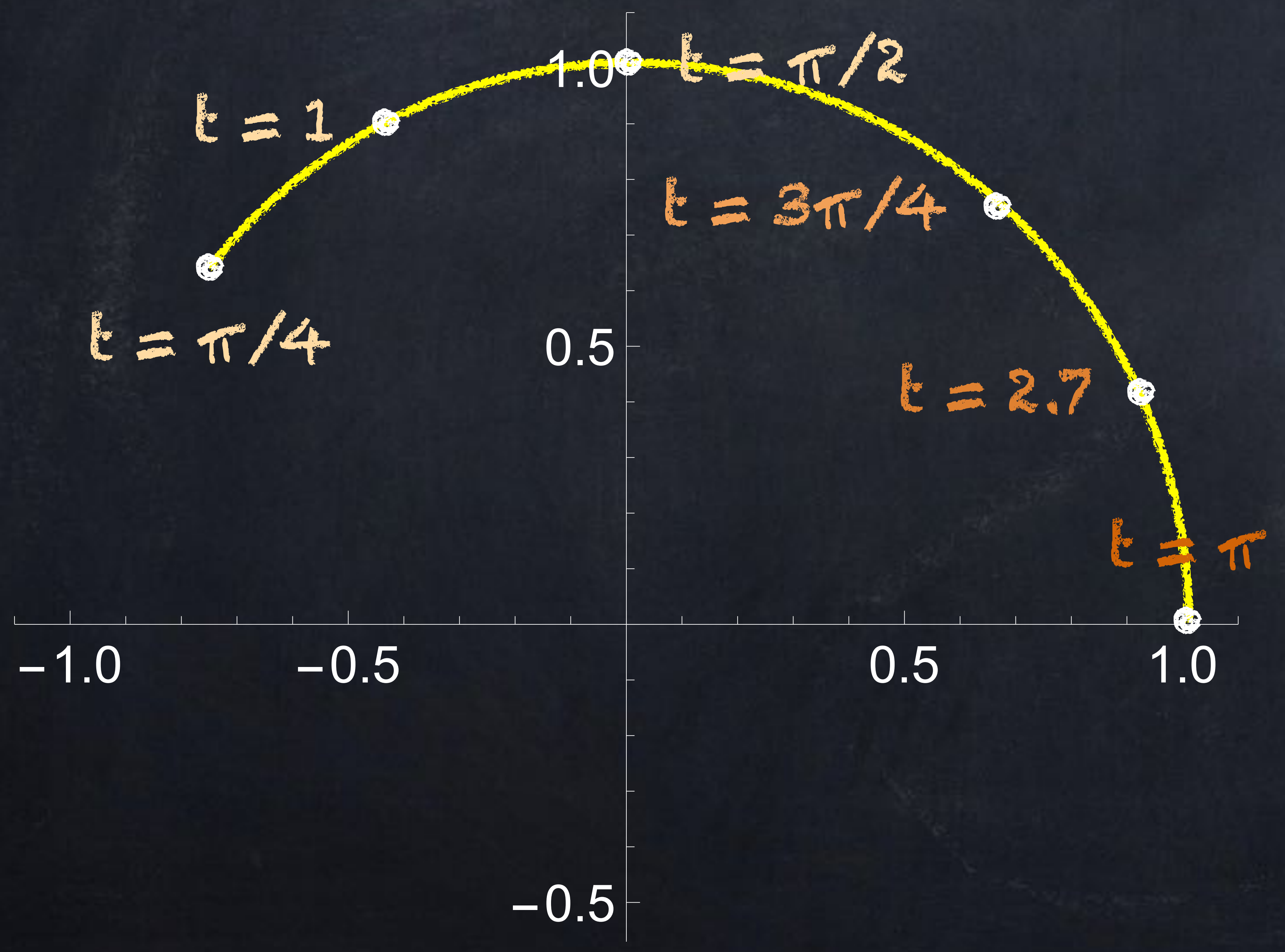
Last week, I suggested using the following formula:

$$\int_a^b f(\vec{r}(t)) \left| \vec{r}'(t) \right| \mathrm{d}t.$$

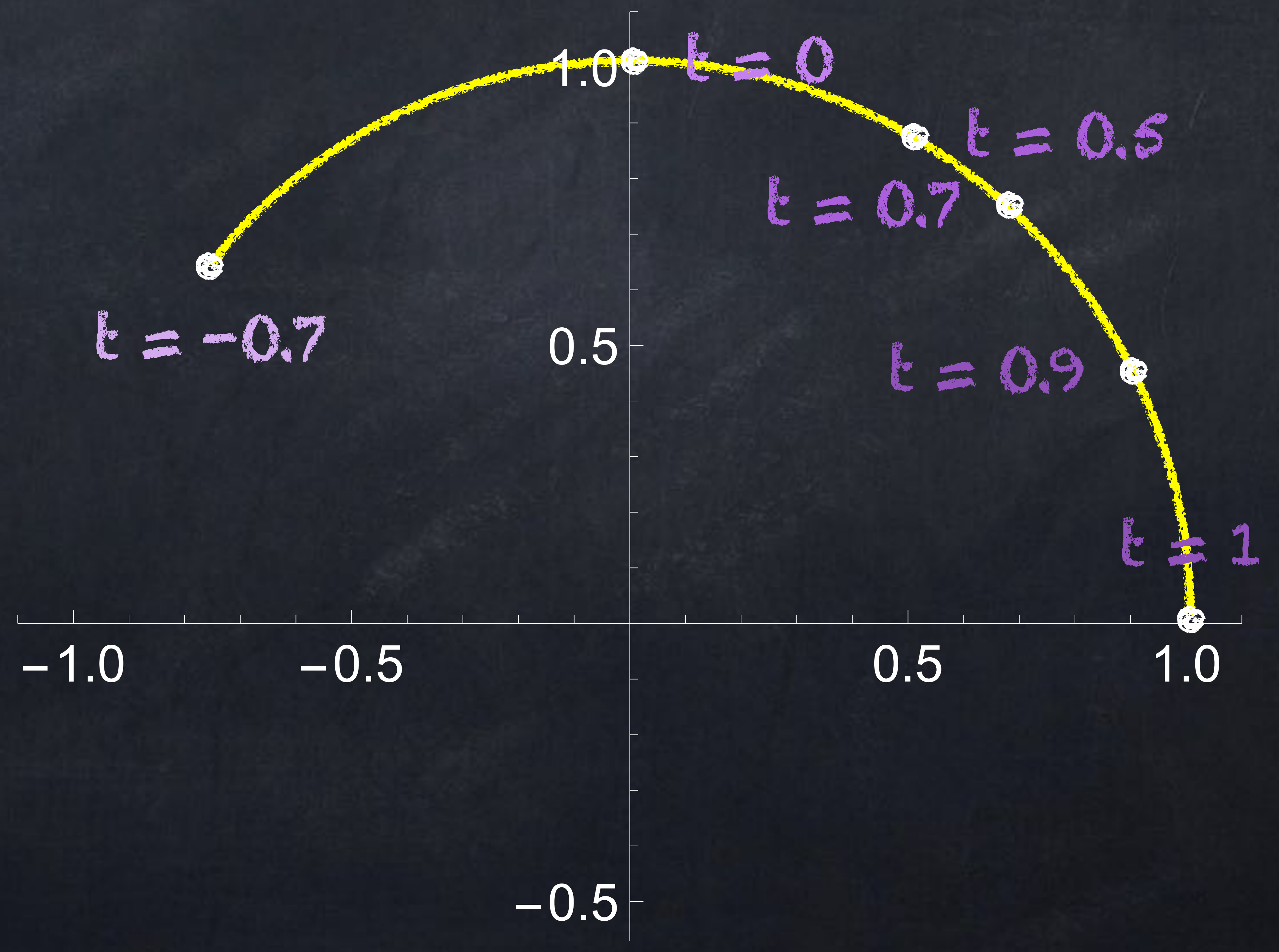
even though $\vec{r}$ isn't actually part of the definition above.

In the warm-up, we used three different $\vec{r} : [a, b] \to \mathbb{R}^2$ to compute three path integrals, but two of them had equal values. This was not a coincidence.

$x = -\cos(t),\ y = \sin(t)$
with $\frac{\pi}{4} \leq t \leq \pi.$



$x = t,\ y = \sqrt{1 - t^2}$
with $\frac{-1}{\sqrt{2}} \leq t \leq 1.$



Fact: If $\vec{r} = [x, y]$ with $a \leq t \leq b$

and

$\overrightarrow{R} = [x, y]$ with $c \leq t \leq d$

are two different **parameterizations** of the same curve, then

$$\int_a^b f(\vec{r}(t)) \left| \vec{r}'(t) \right| \mathrm{d}t \quad \text{and} \quad \int_c^d f(\overrightarrow{R}(t)) \left| \overrightarrow{R}'(t) \right| \mathrm{d}t$$

will be equal.

This is why we can talk about "$\int_C f \, \mathrm{d}s$" for a curve!

Example task: “Integrate x^3y over the clockwise arc of the circle $x^2 + y^2 = 1$ with $x \geq \frac{-1}{\sqrt{2}}$ and $y \geq 0$.”

This was exactly the warm-up for students at the beginning or end of the alphabet (the middle group’s $\vec{r}$ was for a different curve).

The answer is $\frac{1}{16}$. You *could* use either

- $x = -\cos(t),\ y = \sin(t),\ \frac{\pi}{4} \leq t \leq \pi$

or

- $x = t,\ y = \sqrt{1 - t^2},\ \frac{-1}{\sqrt{2}} \leq t \leq 1$

to do this path integral, but the first choice is much easier.

Task 2: "Integrate $\sqrt{2y - x}$ over the line segment from $(0,0)$ to $(1,3)$."

- Step 1: Come up with parametric equations (or a vector equation) to describe this line segment.
- Step 2: Use the formula $\int_a^b f(\vec{r}(t)) \left| \vec{r}'(t) \right| \mathrm{d}t$.

$$= \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2} \mathrm{d}t$$

Answer: $\frac{10}{3}\sqrt{2}$

Parametric equations

In this course you will only have to create your own equations for three kinds of paths:

- straight line segments,
- arcs of circles,
- combinations of line segments and arcs (just *add* the path integrals over each part of the complex path).

For other kinds of curves, an $\vec{r}$ equation (or separate $x =$ ..., $y =$ ... equations) will be given in the task.

Parametric equations

In this course you will only have to create your own equations for three kinds of paths:

- straight line segments,

  The line from point $\overrightarrow{A}$ to point $\overrightarrow{B}$ can always be parameterized as

  $$\vec{r} = (1-t)\overrightarrow{A} + t\overrightarrow{B}$$

  with $0 \leq t \leq 1$, although sometimes other choices are easier.

- arcs of circle,

  The arc of a circle of radius R centered at (h, k) is always

  $$x = h + R\cos(t), \qquad y = k + R\sin(t)$$

  with the bounds for t depending on which part of the circle is used.

From Analysis 1 you should know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[g(t)\cdot\text{constant}\right] = \frac{\mathrm{d}g}{\mathrm{d}t}\cdot\text{constant}$$

and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)+g(t)\right] = \frac{\mathrm{d}f}{\mathrm{d}t}+\frac{\mathrm{d}g}{\mathrm{d}t}.$$

If the “constants” are actually vectors, this still works. So, for example,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(t^5)\hat{\imath}+\ln(t)\hat{\jmath}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(t^5)\hat{\imath}\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(t)\hat{\jmath}\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(t^5)\right]\hat{\imath} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(t)\right]\hat{\jmath}\\ &= \frac{5t^4}{t^5}\hat{\imath}+\frac{1}{t}\hat{\jmath}\end{aligned}$$

Last week

Last week

From Analysis 1 you should know that

$$\big(g(t)\cdot\text{constant}\big)' = g'(t)\cdot\text{constant}$$

and that

$$\big(g(t)+h(t)\big)' = g'(t)+h'(t).$$

If the “constants” are actually vectors, this still works. So, for example,

$$\begin{aligned}\big(\ln(t^5)\hat{\imath}+\ln(t)\hat{\jmath}\big)' &= \big(\ln(t^5)\hat{\imath}\big)' + \big(\ln(t)\hat{\jmath}\big)' \\ &= \big(\ln(t^5)\big)'\hat{\imath} + \big(\ln(t)\big)'\hat{\jmath} \\ &= \frac{5t^4}{t^5}\hat{\imath} + \frac{1}{t}\hat{\jmath}\end{aligned}$$

Last week

From algebra, you should know the vector symbols

$$\hat{\imath} = [1,0] \qquad \text{and} \qquad \hat{\jmath} = [0,1]$$

in 2D (in 3D we have $\hat{\imath}, \hat{\jmath}, \hat{k}$) and you should know how to calculate the length of a vector.

Combining all of this, if

- $\vec{r} = \ln(t^5)\hat{\imath} + \ln(t)\hat{\jmath}$

then we know

- $\left|\vec{r}\,'\right| = \left|\frac{5}{t}\hat{\imath} + \frac{1}{t}\hat{\jmath}\right| = \sqrt{\left(\frac{5}{t}\right)^2 + \left(\frac{1}{t}\right)^2} = \frac{\sqrt{26}}{t}.$

Partial derivatives

Last time

For a function with multiple inputs we can change x or change y (or both at once—more on that later), so we have multiple ways to take derivatives.

The **partial derivative of $f(x, y)$ with respect to x** can be written as any of

$$f'_x(x, y) \qquad f'_x \qquad D_x f \qquad \partial_x f \qquad \frac{\partial f}{\partial x}$$

and is what you get if you *think of every letter other than x as a constant*. Like with $f'(x)$ and $f'(a)$ from An. 1, we also have the **partial derivative of f with respect to x at the point (a, b)**, which is a single number; we write this as $f'_x(a, b)$.

There is also a partial derivative with respect to y (and to z if there are 3 inputs).

Task 1: Calculate $\frac{\partial}{\partial x}\left[y^2\sin(x)\right]$. This is f'_x for the function $f(x, y) = y^2\sin(x)$.

Task 2: Calculate $\frac{\partial}{\partial y}\left[y^2\sin(x)\right]$. This is f'_y for the function $f(x, y) = y^2\sin(x)$.

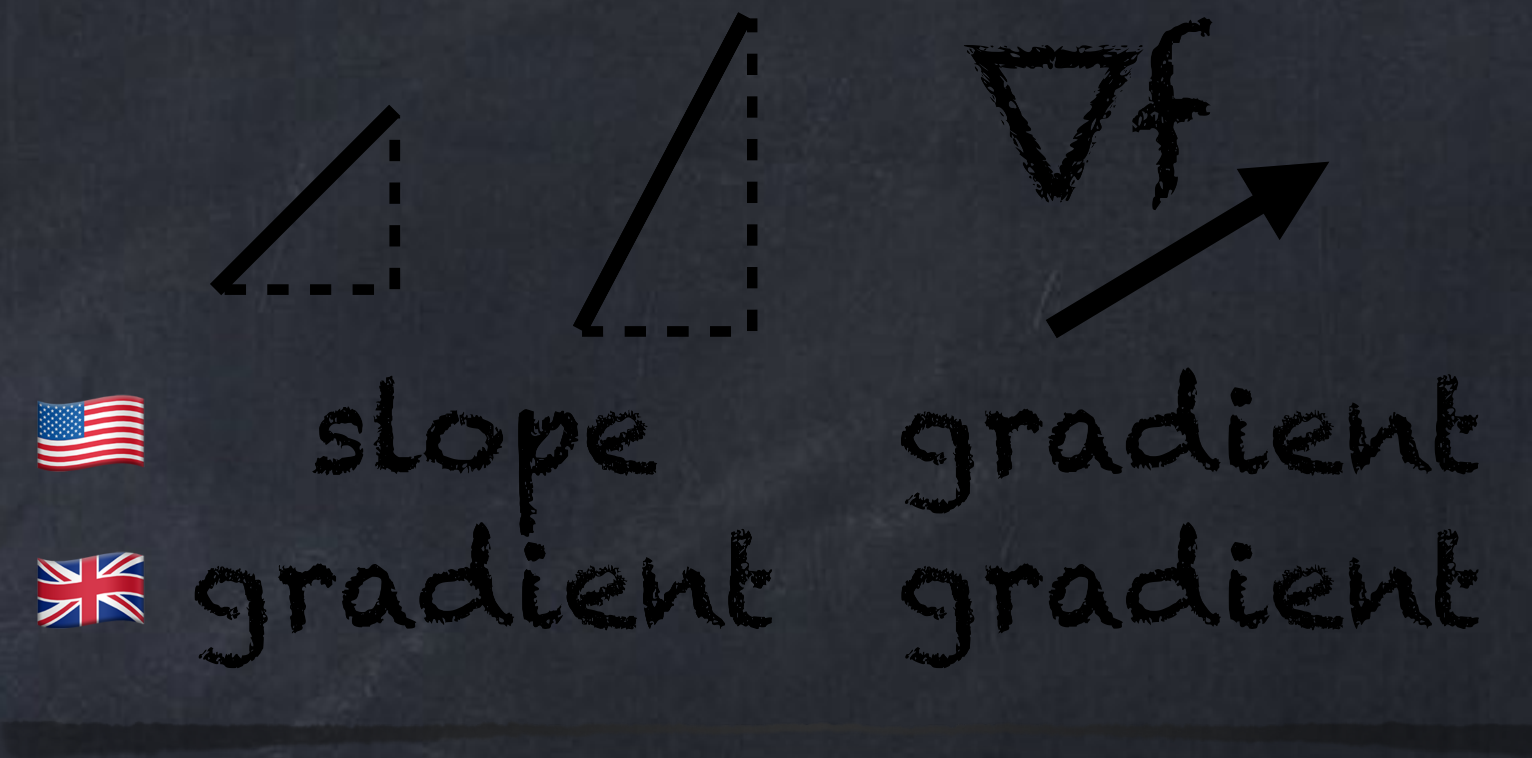
Task 3: Calculate $f'_x(0,3)$ for the function $f(x, y) = y^2 \sin(x)$.

Task 4: Calculate $f'_y(0,3)$ for the function $f(x, y) = y^2 \sin(x)$.

Gradient vector

The **gradient** of the function $f(x, y)$ at the point (a, b) is written $\nabla f(a, b)$ and is the *vector*

$$\nabla f(a,b) = \begin{bmatrix} f'_x(a,b) \\ f'_y(a,b) \end{bmatrix}$$



We can write $\nabla f = [f'_x, f'_y]$ for short.

The gradient function $\nabla f(x, y)$, also written ∇f for short, is a vector that depends on x and y (so it is technically a “vector field”).

Example: Calculate $\nabla f(0,3)$ for the function $f(x, y) = y^2 \sin(x)$.

We already computed $f_x'(0,3) = 9$ and $f_y'(0,3) = 0$, so we just combine them into the vector $[9, 0]$ or $9\hat{\imath}$.

Another way to do this is to think of $\nabla f(x,y)$ as the vector-with-formulas

$$\nabla f = \begin{bmatrix} y^2\cos(x) \\ 2y\sin(x) \end{bmatrix}$$

and then plug in $x=0$, $y=3$ to get $\nabla f(0,3) = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$.

Partial derivatives

Given a function $f(x, y)$, we can calculate

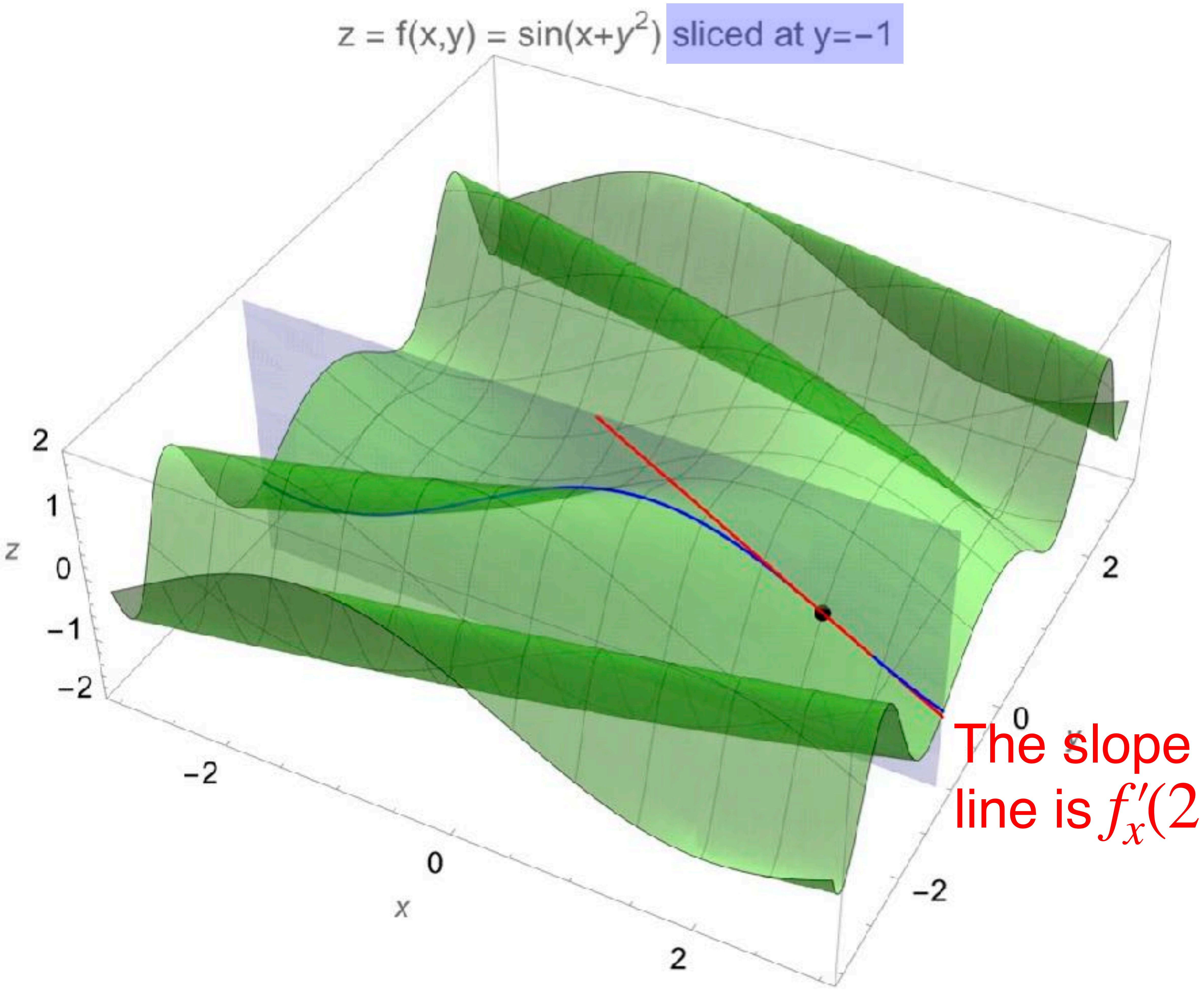
- the function $f_x'(x, y)$
- the function $f_y'(x, y)$
- the number $f_x'(8, 5)$
- the number $f_y'(8, 5)$
- the vector $\nabla f(8, 5) = \begin{bmatrix} f_x'(8, 5) \\ f_y'(8, 5) \end{bmatrix}$.

The "$(8, 5)$" could be any point; the coordinates 8 and 5 are just an example.

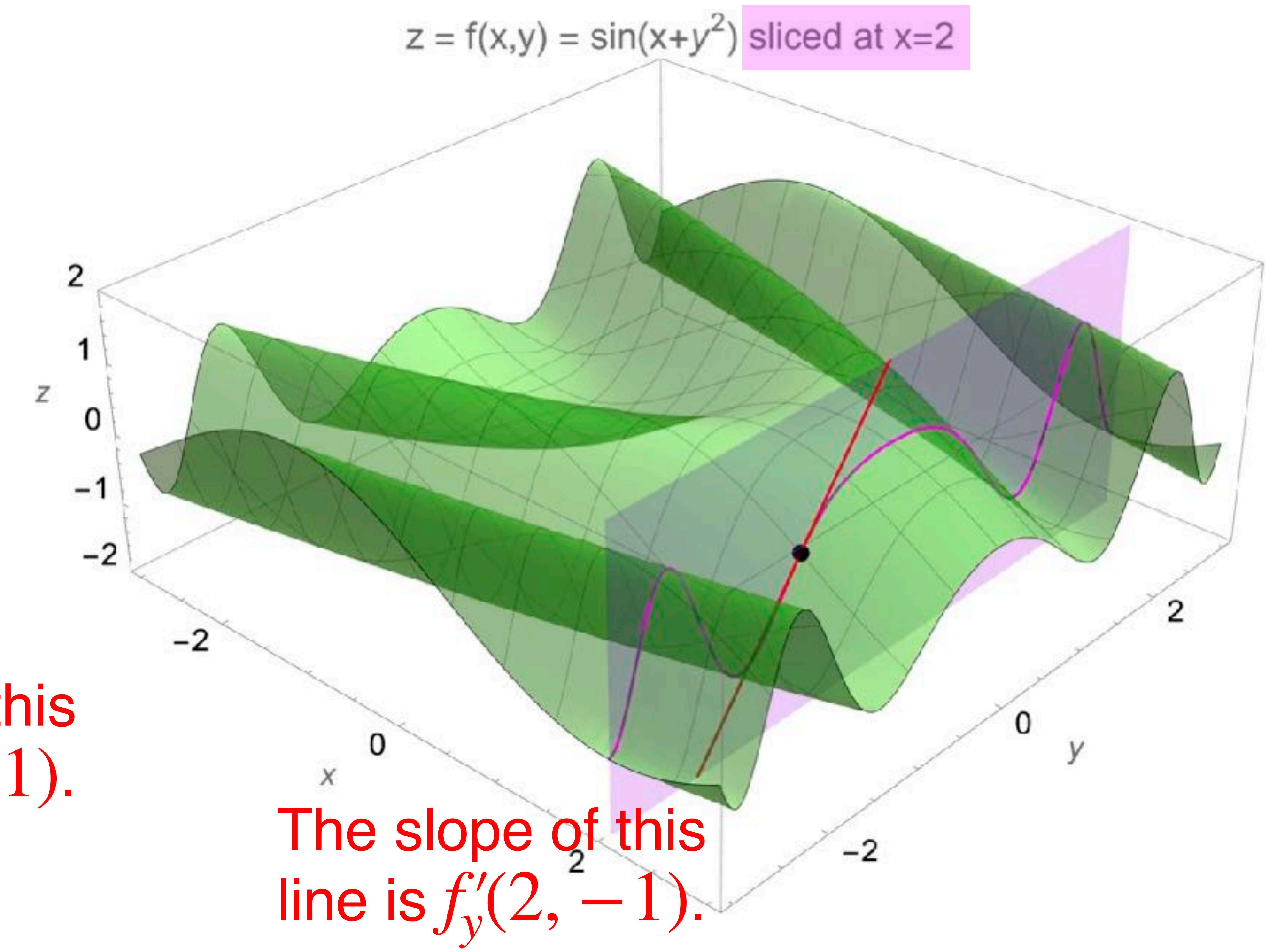
What do these mean??

Like in Analysis 1, we can think of *slope* of some tangent line or we can think of "*rate of change*" more generally.

- For the slope, we need to think about what keeping other variables constant means visually. Note that, for example, $y = -1$ is a plane in 3D space.



The slope of this line is $f'_x(2, -1)$.



The slope of this line is $f'_y(2, -1)$.

Suppose $f(x, y)$ describes the temperature at different positions.
If you stand at (a, b), you have the temperature $f(a, b)$.

- If you move east (right), your temperature changes at the rate $f'_x(a, b)$.
- If you move west (left), your temperature changes at rate $-f'_x(a, b)$.
- If instead you move north (up), your temperature changes at rate $f'_y(a, b)$.
- If instead you move south (down), your temperature changes at rate $-f'_y(a, b)$.

- What if you move northeast? Or south-southwest?

Next week!